Contribution to the study of univariate and multivariate risk processes

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Historical model: for unidimensional risk processes $R_t = u + X_t$,

- with initial reserve $u$
- and with $X_t = ct - S_t$, where
  - $c > 0$ is the premium income rate,
  - $S_t = \sum_{i=1}^{N(t)} W_i$,
  - the $W_i$ are i.i.d. nonnegative random variables, independent from $(N(t))_{t \geq 0}$,
  - with the convention that the sum is zero if $N(t) = 0$.

Probability of ruin: $\psi(u) = P(\exists t \geq 0, R_t < 0)$. 
Two lines of business: classical, 1-dimensional surplus process (black), 2-dimensional process (1 for each line of business, blue and red).

\[ u = u_1 + u_2 \]
Introduction

3 main directions:

- How to model the stochastic dependence between the $K$ lines of business?
- Which ruin concept?
- Within finite or infinite time?
- Ruin or severity of ruin?
- Optimal initial reserve allocation?
- How to measure risk and profit (dividends)?
PhD thesis based on 5 papers:


Structure of the exposé

Contribution to the study of univariate and multivariate risk processes

- Risk and profit measures for univariate risk processes
- A general optimal reserve allocation strategy
- A multidimensional risk model
- Time to ruin, dividends and insolvency penalties
Risk and profit measures for univariate risk processes

- A general optimal reserve allocation strategy
- A multidimensional risk model
- Time to ruin, dividends and insolvency penalties
• the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$,

• the severity of ruin $|u + X_{T_u}|$, or the couple $(T_u, |u + X_{T_u}|)$,

• the time in the red (below 0) from the first ruin to the first time of recovery $T_u' - T_u$, where

$$T_u' = \inf\{t > T_u, u + X_t = 0\},$$

• the maximal ruin severity $(\inf_{t>0} u + X_t)$,

• the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T_u'} |u + X_t| \, dt$,

• the total time in the red $\tau(u) = \int_0^{+\infty} 1\{u + X_t < 0\} \, dt$. 
Risk measures

- the time to ruin $T_u = \inf \{ t > 0, u + X_t < 0 \}$,
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Stéphane Loisel, 12/2004
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- the maximal ruin severity $(\inf_{t>0} u + X_t)$,
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- the total time in the red $\tau(u) = \int_0^{\infty} 1\{u + X_t < 0\} \, dt$. 
Consider risk measures based on some fixed time interval \([0, T]\) 
(T may be infinite).

Simple penalty function (expected penalty to pay due to insolvency until time horizon \(T\)):

\[
\mathbb{E} \left( I_T(u) \right) = \mathbb{E} \left( \int_0^T 1_{\{u+X_t<0\}} |u + X_t| \, dt \right).
\]

Note that the probability \(\mathbb{P} \left( I_T(u) = 0 \right)\) is the probability of non ruin within finite time \(T\).
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$$

Note that the probability $\mathbb{P}(I_T(u) = 0)$ is the probability of non ruin within finite time T.
Problem: in the compound-Poisson risk model, with claim amounts taking values in $\mathbb{N}$, compute the probability of non ruin before finite time $T$ defined by

$$\phi(u, T) = \mathbb{P} \left( \forall t \in [0, T], \ R_t \geq 0 \right).$$

The Picard-Lefèvre formula is based on generalized Appell polynomials. The Seal-type formula is based on a lemma from Tákacs and on sample path properties.

Proof of the Picard-Lefèvre formula based on sample path properties, and class of new formulae (Rullière and L., 2004a)

Drawback: very hard to generalize the Seal-type formula in a multi-risks model.
The two formulae may be written with the notation $h_j(\tau) = \tilde{P}[S_{\tau/c} = j]$:

$$\phi_{PL}(u, t) = \sum_{j=0}^{u} \left[ h_j(t) + h_j(j-u) \sum_{i=u+1}^{u+n} h_{i-j}(u+t-j) \frac{u+t-i}{u+t-j} \right]$$

$$\phi_0(u, t) = \sum_{j=0}^{u+n} h_j(t) - \sum_{k=1}^{n} h_{u+k}(k) \sum_{l=0}^{n-k} h_{n-k-l}(t-k) \frac{t-k-l}{t-k}$$

The equivalence may be proved thanks to pseudo-compound Poisson distributions and the following result:

**Lemma 1** (Rullière and L., 2004a) For $t \in \mathbb{N}^*$, $x \in \mathbb{N}$ and for all $z \in \mathbb{Z}$, $z > x - t$,

$$\mathbb{P}(S(t) = x) = \sum_{j=0}^{x} \mathbb{P}(S(z+t-j) = x-j) \tilde{P}(S(j-z) = j) \frac{z+t-x}{z+t-j}. \quad (1)$$
From an economical point of view, it seems more consistent to consider

\[
\mathbb{E} I_{g,h}(u) = \mathbb{E} \left( \int_0^T \left( 1_{\{u + X_t \geq 0\}} g(|u + X_t|) - 1_{\{u + X_t \leq 0\}} h(|u + X_t|) \right) dt \right)
\]

- \(0 \leq g \leq h\)
- \(g\) corresponds to a reward function for positive reserves,
- and \(h\) is a penalty function in case of insolvency.

These risk measures may be differentiated with respect to the initial reserve \(u\).

Fubini’s theorem.
Other profit indicator: dividends paid until ruin.

Horizontal barrier strategy for dividend payment (at level $b$):
modified surplus process $U_b(t)$ (red) and dividend process $L(t)$ (blue).

$P(L(T_u) > 0)$ is the probability to reach $u + (b - u)$ from $u$ before ruin

- $\rightarrow$ win first probability (Rullière and L., 2004b): $WF(u, v) = P(T_u > T_{uw})$,
- where $T_u = \inf \{t, R_t < 0\}$ and $T_{uw} = \inf \{t, R_t \geq u + v\}$.
- Property: For $v, w \geq 0$, $WF(u, v + w) = WF(u, v)WF(u + v, w)$. 
\[
\Theta = \sup \{ R_t, t \leq T_0 \mid R_0 = 0 \}.
\]

For \( u, v \geq 0 \),
\[
WF(u, v) = \mathbb{P}(\Theta \geq u + v \mid \Theta \geq u).
\]

Hazard rate of \( \Theta \):
\[
\mu_u(v) = -\frac{\partial}{\partial v} \ln WF(u, v).
\]

This rate is finite, only depends on \( u + v \), and may be written \( \mu_u(v) = \mu_{u+v} \).

Algorithm to compute the hazard rate function of \( \Theta \) and its derivatives. Efficient way to obtain \( WF(u, v) \) and its derivatives numerically (Rullière and L., 2004b).
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Differentiation theorems

\[ I(u) = \int_0^T 1_{\{u + X_t < 0\}} |u + X_t| \, dt \]
Th. (L., 2004b):

Let \((X_t)_{t\in[0,T]}\) be a stochastic process with almost surely time-integrable sample paths. For \(u \in \mathbb{R}\), denote by \(\tau_0(u)\) the time spent in zero by the process \(u + X_t\):

\[
\tau_0(u) = \int_0^T 1_{\{u + X_t = 0\}} \, dt.
\]

Let \(f\) be defined by \(f(u) = \mathbb{E}(I_T(u))\) for \(u \in \mathbb{R}\), where

\[
I_T(u) = \left( \int_0^T 1_{\{u + X_t < 0\}} |u + X_t| \, dt \right).
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I_T(u) = \left( \int_0^T 1\{u+X_t<0\} |u+X_t| \, dt \right).
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- For \(u \in \mathbb{R}\), if \(\mathbb{E}\tau_0(u) = 0\), then \(f\) is differentiable at \(u\), and \(f'(u) = -\mathbb{E}\tau(u)\).
Th. (L., 2004b): Let $X_t = ct - S_t$, where $S_t$ is a jump process satisfying hypothesis (H1): $S_t$ has a finite expected number of nonnegative jumps in every finite interval, and for each $t$, the distribution of $S_t$ is absolutely continuous. For example, $S_t$ might be a compound Poisson process with a continuous jump size distribution. Define $h$ by $h(u) = \mathbb{E}(\tau(u))$ for $u \in \mathbb{R}$. $h$ is differentiable on $\mathbb{R}^*_+$, and for $u > 0$,

$$h'(u) = -\frac{1}{c} \mathbb{E}N^0(u),$$

where $N^0(u) = \text{Card}\{t \in [0, T], \ u + ct - S_t = 0\}$. 


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h(u) = \mathbb{E}(\tau(u)) \quad \text{for} \quad u \in \mathbb{R}.
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h is differentiable on \( \mathbb{R}^*_+ \), and for \( u > 0 \),
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where \( N^0(u) = \text{Card} \left( \{ t \in [0, T], \ u + ct - S_t = 0 \} \right) \).

This remains valid with \( T = +\infty \) if \( X_t \) has a positive drift and \( \tau(u) \) is integrable. In the compound Poisson case, for \( u \geq 0 \),
\[
h'(u) = -\frac{1}{c} \frac{1}{1 - \psi(0)} \psi(u)
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where \( N_0^0(u) = \text{Card} (\{ t \in [0, T], \quad u + ct - S_t = 0 \}) \).

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\( \mathbb{E}I_T(.) \) is thus well strictly convex, which will be very important for minimization.
Theorem: In the Poisson-Exponential$(1/\mu)$ case, $\psi(u) = \frac{1 - \mu R}{\mu R} e^{-R u}$, with $R = \frac{1}{\mu} \left(1 - \frac{\lambda \mu}{c}\right)$. Hence, for $T = +\infty$,

\[ \mathbb{E}\tau(u) = \frac{1 - \mu R}{c \mu R^2} e^{-R u} \quad \text{Gerber, Dos Reis (1993)} \]

and

\[ \mathbb{E}I_\infty(u) = \frac{1 - \mu R}{c \mu R^3} e^{-R u} \quad \text{L. (2004b)} \]

Proof: Integration of the well-known formula for $\psi(u)$. The considered functions tend to 0 as $u \to +\infty$.

It is possible to derive $\mathbb{E}I_\infty(u)$ explicitly for $\Gamma$ and phase-type-distributed claim amounts.
Th. (L., 2004b): Let \( g, h \) be two convex or concave functions in \( C^1(\mathbb{R}^+, \mathbb{R}^+) \), such that for \( x \geq 0, g(x) \geq g(0) \) and \( h(x) \geq h(0) \). Let \( X_t \) be a stochastic process such that \( t \to g(u + X_t) \) and \( t \to h(u + X_t) \) are almost surely integrable on \([0, T]\). Let \( I_g^+ \) be the function from \( \mathbb{R} \) into the space of nonnegative random variables, and defined by

\[
I_g^+ (u) = \int_0^T \1_{\{u + X_t \geq 0\}} g(u + X_t) \, dt
\]

for \( u \geq 0 \) and let \( f(.) = \mathbb{E}I_g^+(.) - \mathbb{E}I_h(.) \). Define also

\[
L_T(0) = \lim_{\varepsilon \downarrow 0} \left( \frac{1}{2\varepsilon} \int_0^T \1_{\{\|u + X_t\| < \varepsilon\}} \, dt \right).
\]

If, for \( u \in \mathbb{R}, \quad \mathbb{E}I_g^+(u), \quad \mathbb{E}I_h(u), \quad \mathbb{E}I_g'(u), \quad \mathbb{E}I_h'(u) < +\infty, \)
and if \( \mathbb{E}T_0(u) = 0 \), then \( f \) is differentiable on \( \mathbb{R}_+^* \), and for \( u > 0 \),

\[
f'(u) = \mathbb{E}I_g'(u) - \mathbb{E}I_h'(u) - (g(0) + h(0))\mathbb{E}L_T(0)
\]
What has to be minimized is

$$A(u_1,\ldots,u_K) = \sum_{k=1}^{K} \mathbb{E}I^k_T(u_k)$$

under the constraint $u_1 + \cdots + u_K = u$, where

$$\mathbb{E}I^k_T(u_k) = \mathbb{E} \left[ \int_0^T |R^k_t| 1\{R^k_t < 0\} \, dt \right]$$

with $R^k_t = u_k + X^k_t$
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This does not depend on the dependence structure.
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\[ \mathbb{E} I^k_T(u_k) = \mathbb{E} \left[ \int_0^T \left| R^k_t \right| 1_{\{R^k_t < 0\}} dt \right] \]

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From previous differentiation theorems, \( A \) is strictly convex. On the compact space

\[ S = \{(u_1, \ldots, u_K) \in (\mathbb{R}^+)^K, \; u_1 + \cdots + u_K = u\}, \]

\( A \) admits a unique minimum.
Lagrange multipliers \(\rightarrow\) optimal allocation:

there is a subset \(J \subset [1, K]\) such that

- for \(j \notin J\), \(u_j = 0\),
- and for \(j, k \in J\), \(\mathbb{E} \tau_j = \mathbb{E} \tau_k\).
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In the Poisson-Exponential(\( \frac{1}{\mu} \)) case, recall that

\[
E I_u = 1 - \frac{\mu R}{c\mu R^3} e^{-Ru}.
\]

Consider a two-line model, with the following parameters:

\( \mu_1 = \mu_2 = 1 \), \( c_1 = c_2 = 1 \), \( R_2 = 0.4 \) and \( u = 10 \).

Three values of \( R_1 \) \( \rightarrow \) different optimal allocation strategies.
Figure 4: Graph of $A(x, 10 - x)$.
When $R_1 = 0.5 > R_2$,
line of business 1 is safer than line 2.
→ $u_1 < u_2$.
The optimal allocation is about
$(u_1 = 3.5, \ u_2 = 6.5.)$

Figure 5: Graph of $A(x, 10 - x)$.
When $R_1 = 0.08 < R_2$,
line of business 1 is much riskier than line 2.
→ $u_1 = u = 10$ and $u_2 = 0$.
Transfer of the whole reserve to line 1.
Contributions to the study of univariate and multivariate risk processes

- Risk and profit measures for univariate risk processes
- A general optimal reserve allocation strategy
- A multidimensional risk model
- Time to ruin, dividends and insolvency penalties
  – in discrete time
  – with independent increments
  – increments follow any distribution of $\mathbb{Z}^K$ or $\mathbb{R}^K$.

• In our model, two main kinds of phenomena:
  – Claims for different lines of business may come from a common event:
    simultaneous jumps for the multivariate process $\rightarrow$ Poisson common shock model.
  – Markovian environment:
    Modulation by a Markov process which describes the evolution of the state of the environment (Asmussen, 1989).
$n$ states of the environment and $K$ lines of business

- State of the environment $\rightarrow$ Markov process $J(t)$
  - with initial distribution $\pi_0$
  - and rate transition matrix $Q$.
- Claim amounts: for $1 \leq i \leq n$, sequence of i.i.d. random vectors $(W_i^m)_{m \geq 1}$
  - taking values in $(\mathbb{R}^+)^K$, with distribution function $F_{W_i}$
  - such that if a claim hits line $k$, amount for line $k$ exponentially distributed,
  - and independent from a Poisson process $N_i^i(t)$ with parameter $\lambda^i$.
- Define the $n$ independent $K$-dimensional Lévy processes
  \[
  X^i(t) = c^i t - \sum_{m=1}^{N_i^i(t)} W^i_m
  \]
Then, define \( X(t) = (X_1(t), \ldots, X_K(t)) \) as follows:

let \( T_p \) be the instant of the \( p^{th} \) jump of the process \( J_t \), and

\[
\forall k \leq K, \quad X(t) - X(0) = \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X_i(T_p) - X_i(T_{p-1})) \mathbb{1}_{\{J_{T_p-1} = i, T_p \leq t\}} \\
+ \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X_i(t) - X_i(T_{p-1})) \mathbb{1}_{\{J_{T_p-1} = i, T_{p-1} \leq t < T_p\}}.
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\[
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let \( T_p \) be the instant of the \( p^{th} \) jump of the process \( J_t \), and

\[
\forall k \leq K, \quad X(t) - X(0) = \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X^i(T_p) - X^i(T_{p-1})) 1\{J_{T_{p-1}} = i, T_p \leq t\}
\]

\[
+ \sum_{p \geq 1} \sum_{1 \leq i \leq n} (X^i(t) - X^i(T_{p-1})) 1\{J_{T_{p-1}} = i, T_{p-1} \leq t < T_p\}.
\]
Dependence structure between risks

Theorem 1

\[ M'(t, \alpha) = e^{-\langle \alpha, X(t) \rangle} \tilde{1}_t e^{-F(\alpha_1, \ldots, \alpha_K) t} \]

is a \( n \)-dimensional martingale (\( n \) is the environment state space size) for all \( \alpha \in \mathbb{C}^K \) such that the \( \phi_k^i(\alpha_k) \) all exist, and for all distribution of \( (X(0), J_0) \), where

\[ F(\alpha_1, \ldots, \alpha_K) = Q - \text{diag}(\phi^1(\alpha_1, \ldots, \alpha_K), \ldots, \phi^n(\alpha_1, \ldots, \alpha_K)). \]

If \( h(\alpha_1, \ldots, \alpha_K) \) is a right eigenvector of \( F(\alpha_1, \ldots, \alpha_K) \) with eigenvalue \( \lambda(\alpha_1, \ldots, \alpha_K) \), then

\[ N'(t, \alpha) = e^{-\langle \alpha, X(t) \rangle} e^{-\lambda(\alpha_1, \ldots, \alpha_K) t} h J_t (\alpha_1, \ldots, \alpha_K) \]

is a martingale.

- not convenient to apply Doob’s optimal stopping theorem directly.
- Discretize time and space
  - Compute numerically finite-time ruin probabilities (L., 2004a).
    - Ruin: the multivariate claim process enters an insolvency region.
    - Algorithm involves generalized Appell functionals (Picard et al, 2003).
Contribution to the study of univariate and multivariate risk processes

- Risk and profit measures for univariate risk processes
- A general optimal reserve allocation strategy
- A multidimensional risk model
- Time to ruin, dividends and insolvency penalties
• The model we propose takes into account dependence between lines of business both for
the multivariate claim process, and for the premium incomes and dividends.

• We consider that line of business 1 behaves slightly differently from the other ones (it
might correspond for example to a main company with $K - 1$ subcompanies).

• Dependence between claim arrivals and amounts of the $K$ lines of business
  ➔Previous model with common shocks in a Markovian environment.

• For each line $k \geq 1$, upper barrier $b_k$ ➔dividends.

• Financial interactions: positions of subcompanies influence the premium income rate of
  line 1.

• Cause: the surplus of the main company may be partly invested in some subcompanies.
Surplus process of the main company (line 1) and of one subcompany (line 2): the drift for line 1 is a function of the surplus of line 2.
Surplus process of the main company (line 1) and of one subcompany (line 2): the drift for line 1 is a function of the surplus of line 2.
Surplus process of the main company (line 1) and of one subcompany (line 2): the drift for line 1 is a function of the surplus of line 2.
Surplus process of the main company (line 1) and of one subcompany (line 2): the drift for line 1 is a function of the surplus of line 2.
Typical questions that arise are:

• what proportion of the excess of the surplus of line of business 1 is in average lost for the shareholders due to the insolvency of some other lines of business? This represents for the shareholders a loss of dividends that they would not have undergone if the subcompanies were completely separate.

• Does the expected time to ruin of a line of business increase or decrease due to the possible financial support or penalty coming from the impact of the surpluses of other lines of business?

• What is the probability for a line of business to get recovered after its ruin?
Horizontal barrier strategy for dividend payment (at level $b$):
modified surplus process $U_b(t)$ (red) and dividend process $L(t)$ (blue).

$$Z(t) = b - U_b(t).$$

\[ M(t, \alpha) = \int_{0}^{t} e^{\alpha Z(s)} \mathbf{1}_{J(s)} ds \mathbf{K}(\alpha) + e^{\alpha Z(0)} \mathbf{1}_{J(0)} - e^{\alpha Z(t)} \mathbf{1}_{J(t)} + \alpha \int_{0}^{t} \mathbf{1}_{J(s)} dL(s) \]

(2)

is a \( n \)-dimensional martingale for all \( \alpha \in \mathbb{C} \) such that the \( \phi^j_k(\alpha) \) exist and for all distributions of \( (X(0), J_0) \).

Note that \( \det (\mathbf{K}(\alpha)) \) may be written as a quotient of two polynomials where the numerator is of degree \( 2n \). Assume that the numerator has \( 2n \) distinct roots \( \alpha_1, \ldots, \alpha_{2n} \). Let \( h^j(\alpha_j) \) be a column vector such that \( \mathbf{K}(\alpha_j) h^j(\alpha_j) = 0 \). By multiplying (2) by \( h^j(\alpha_j) \), we get the following system of \( 2n \) equations for the \( p_j = \mathbb{P}(J_\tau = j) \) and \( l_j = \int_{0}^{\tau} 1\{J_s = j\} ds \):

\[ \mathbb{E} \left[ e^{\alpha_j Z(0)} h^j_{J(0)}(\alpha_j) \right] - \sum_{i=1}^{n} p_i e^{\alpha_j b} \frac{1}{1 - \alpha_j \mu_i} h^j_i(\alpha_j) + \alpha_j \sum_{i=1}^{n} l_i h^j_i(\alpha_j) = 0. \]  

(3)
Outline of the method:

• Discretize space for the $K - 1$ subcompanies

• Incorporate the original environment and the position of the $K - 1$ subcompanies to get a new environment.

• Apply results of Frostig (2004) (modified to allow jumps at state change instants) to line 1 in the new environment.

• Use an analogue of Schilder’s theorem for Poisson processes to have almost sure convergence of the sample paths (for the topology of uniform convergence) on all $[0, T]$.

• The time to ruin is less than the time to ruin in the most favorable state, which is integrable.

• Conclude with the dominated convergence theorem.
For line of business $k \geq 2$,
original sample path (black)
and approximated sample path (red) to incorporate the position of line $k$ in the new
environment process.
For line of business $k \geq 2$, original sample path (black) and approximated sample path (red) to incorporate the position of line $k$ in the new environment process.
Conclusion and perspectives

Contribution to the study of univariate and multivariate risk processes

- Risk and profit measures for univariate risk processes
- A general optimal reserve allocation strategy
- A multidimensional risk model
- Time to ruin, dividends and insolvency penalties
- Perspectives
Perspectives:

- More numerical analysis.
- Bound the approximation error in the last model at some fixed step.
- Optimal $b_k, u_k$,
- Further questions about the impact of dependence,
- Link with credit risk theory.
- General case with drifts depending on the positions of all lines of business.
- Take investment into account.
- Estimation of parameters.
Perspectives:

- More numerical analysis.
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http://isfaserveur.univ-lyon1.fr/~stephane.loisel/phd.html