Contributions to quantitative risk management in insurance

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Habilitation à diriger les recherches
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- Dependence between claim amounts influenced by an environment process

2 Optimal reserve allocation

3 Sensitivity and robustness

4 New themes
A recent correlation crisis in Kruger Park

Correlation between short-term mortality indicators can suddenly increase. Here correlation and the marginal risks increase at the same time. Another example: a drive with your mother-in-law.
Classical assumptions and our problem

Classical assumptions: \( R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \)
- claim amounts \((X_i)_{i \geq 1}\): sequence of i.i.d. r.v.’s, with finite mean,
- The \((X_i)_{i \geq 1}\) are independent from \((N(t))_{t \geq 0}\) (e.g. renewal process).

Problem: Derive asymptotics of finite-time ruin probabilities for large risks

\[
\psi(u, t) = P(\exists \tau \in [0, t], R(\tau) < 0 | R(0) = u),
\]

in more general models with dependence between claim amounts and possible correlation crises.
Some types of correlation (not developed in this talk)

Classical assumptions: \( R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \)
  
  - claim amounts \((X_i)_{i \geq 1}\): sequence of i.i.d. r.v.’s, with finite mean,
  - The \((X_i)_{i \geq 1}\) are independent from \((N(t))_{t \geq 0}\) (e.g. renewal process).

Some models with embedded correlations:
  
  - \( c \) is not constant over time and is adjusted to the observed previous claims (with Bühlmann linear credibility premium principle): impact of using credibility theory on the ruin probability (joint work with J. Trufin, 2009)
  
  - dependence between the claim arrival process and the claim sizes (earthquake risk, flooding and drought risk, …). Works of Boudreault et al. (2006), Albrecher et al. (2007), joint work with R. Biard, C. Lefèvre and H. Nagaraja (2010).
  
  - dependence between claim arrivals and the intensity process: shot-noise processes, cycles influenced by large claims (joint work with M. Bargès and X. Venel, 2009).
Impact of dependence between claim amounts

- In practice, claim amounts are influenced by common factors.
- This leads to stochastic correlation models.
- This correlation may change during time, due to endogenous risk or external shocks (joint works with Wayne Fisher and Shaun Wang (2007) and with Pierre Arnal and Romain Durand (2010)), parameter uncertainty (see Meyers (1999)), ...
- What happens
  - if dependence between claim amounts is governed by a Markovian environment process?
  - if claim amounts of different lines of business suddenly become more dependent (in a common shock model)?
  - if those correlation crises are triggered by some large claims?
Systemic risk: securitization

Does securitization really atomize risk?
Systemic risk: securitization

Risk is just transferred and recombined, but does not disappear. If a large risk becomes reality, all counterparts may be affected and/or downgraded. Reinsurer’s default can lead to sudden increase in frequency and correlation of claim amounts for the insurer. How to compute the ruin probability in that case? Work in progress with C. Blanchet, D. Dorobantu and S. Louhichi.
How can independent risks become suddenly strongly correlated?

Endogenous uncertainty:

- Uncertainty is generated/modified by response of individual entities to events
- Feedback loop: outcomes $\rightarrow$ forecasts $\rightarrow$ decisions $\rightarrow$ outcomes $\rightarrow$ revised forecasts $\rightarrow$ revised decisions $\rightarrow$ … (Millennium Bridge)
- Statistical relationships are endogenous to the model, and may undergo structural shifts (Goodhart’s Law: Any observed statistical regularity will tend to collapse once pressure is placed upon it for control purposes)
- Relevant when individual entities are similar in terms of forecasts and likely reactions to events
- Relevant when outcomes are sensitive to concerted actions
How can independent risks become suddenly strongly correlated?

**Endogenous Risk**
- Risk from shocks generated and amplified within the system

in contrast to...

**Exogenous Risk**
- Risk from shocks from outside the system

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**Millennium Bridge**
- New design
- Tested with extensive simulations
- All angles covered
- No endogenous shocks

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**What Endogeneity?**
- Pedestrians had some problems
- Bridge closed
How can independent risks become suddenly strongly correlated?

**Diagnosis**

- Trouble at 1 hertz (one complete cycle per second)
- Walking pace: two steps per second (2 hertz)
- But sideways motion every two steps (1 hertz)
How can independent risks become suddenly strongly correlated?

Bridge moves $\rightarrow$

Further adjust stance

$\uparrow$

Adjust stance $\downarrow$

$\leftarrow$ Push bridge
Analogy with LTCM

Feedback Revisited

Margin calls →

↑

Distress

Unwind leveraged trades

↓

← Adverse price move

This analogy is drawn from Danielsson and Shin (2003). Similar potential feedback loops for surrender risk in life insurance.
Focus on some heavy-tailed distributions

**Definition (Regular variation class (\(\mathcal{R}\))**

\(F\) belongs to \(\mathcal{R}_{-\alpha}, \alpha \geq 0\) if and only if

\[
\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha}
\]

for any \(y > 0\). Note that if \(0 < \alpha < 1\), any \(X\) with c.d.f. \(F\) has infinite mean.

**Definition (Multivariate Regular variation class (\(\mathcal{MR}\))**

A random vector \(X = (X_1, \ldots, X_n)\) belongs to \(\mathcal{MR}_{-\alpha}, \alpha > 0\) if and only if there exists a \(\theta \in \mathbb{S}^{n-1}\), where \(\mathbb{S}^{n-1}\) is the unit sphere with respect to a norm \(|\cdot|\); such that

\[
\frac{P(|X| > tu, X/|X| \in \cdot)}{P(|X| > u)} \overset{\nu}{\to} t^{-\alpha} P_{\mathbb{S}^{n-1}} (\theta \in \cdot),
\]

where \(\overset{\nu}{\to}\) denotes vague convergence on \(\mathbb{S}^{n-1}\).
A basic situation

∀n ≥ 1, X_n = I_nW_0 + (1 - I_n)W_n,

where

- \((W_n)_{n \geq 0}\) : i.i.d. sequence, common cdf \(F_W \in \mathcal{R}_{-\alpha}, \alpha \geq 0\),
- \((I_n)_{n \geq 1}\) is a sequence of i.i.d Bernoulli random variables, with \(P(I_1 = 1) = p \in [0, 1]\),

- The \(W_n, n \geq 0\) are independent from the \(I_k, k \geq 1\).

Denote the aggregate claim amount (with \(S^p_t = 0, \text{if } N_t = 0\)) by

\[S^p_t = \sum_{k=1}^{N(t)} X_k.\]

First, we calculate \(P(S^p_t > x)\) for large \(x\), then we approximate \(\psi^p(u, t)\) by \(P(S^p_t > u)\) and, finally, we compare \(\psi^p(u, t)\) and \(\psi^q(u, t)\) for \(0 \leq p < q \leq 1\).
Important properties and consequences

Max-sum property: if $F_1$ and $F_2$ belong to $\mathcal{R}_{-\alpha}$, $\alpha > 0$ then

$$F_1 \ast F_2(x) \sim F_1(x) + F_2(x)$$

for large $x$,

Convolution closure: if $F_1$ and $F_2$ belong to $\mathcal{R}_{-\alpha}$, $\alpha > 0$ then so does $F_1 \ast F_2$.

Since $F_W$ belongs to $\mathcal{R}_{-\alpha}$, for any $k \geq 1$, $1 \leq j \leq k - 1$ and any pairwise distinct $n_1, \ldots, n_{k-j} \geq 1$, for large $x$,

$$P \left( W_{n_1} + \ldots + W_{n_{k-j}} + jW_0 > x \right) \sim (k-j) \overline{F_W}(x) + \overline{F_W} \left( \frac{x}{j} \right)$$

$$\sim \left( k - j + \frac{\overline{F_W} \left( \frac{x}{j} \right)}{\overline{F_W}(x)} \right) \overline{F_W}(x)$$

$$\sim (k - j + j^{\alpha}) \overline{F_W}(x)$$

(1)
\begin{align*}
P(S^p(t) > x) &\sim \left\{ \sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \left[ \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k-j+j^\alpha) \right] \right\} \overline{F_W}(x). \\
\text{can be rewritten as} \quad P[S^p(t) > x] &\sim \left\{ \lambda t + E \left[ (Z^p(t))^\alpha - Z^p(t) \right] \right\} \overline{F_W}(x), 
\end{align*}

(2)

where \( Z^p(t) \) denotes a binomial random variable \( Bin[N(t), p] \).

The mixed binomial law \( MBin[N(t), p] \) is stochastically increasing in \( p \) (see, e.g., Lefèvre and Utev (1996)). Since the function \( f(x) = x^\alpha - x, x \in \{0, 1, \ldots\} \), is decreasing (resp. increasing) when \( \alpha < 1 \) (resp. \( \alpha > 1 \)), we get (for \( u \) large enough),

- for \( \alpha < 1 \) (infinite mean case),
  \[
  0 \leq p < q \leq 1 \Rightarrow \psi^p(u, t) > \psi^q(u, t),
  \]

- and for \( \alpha > 1 \) (finite mean case),
  \[
  0 \leq p < q \leq 1 \Rightarrow \psi^p(u, t) < \psi^q(u, t).
  \]
Correlation crises

Dependence between claim amounts, claim size distribution and intensity are modulated by a Markovian environment process.

- Markov environment process \((J(t))_{t \geq 0}\) with \(J \geq 2\) states 1, \ldots, \(J\)
  - with initial distribution \(\pi_0\)
  - and transition rate matrix \(Q\).

- Claim amounts: for \(1 \leq i \leq n\), sequence of i.d. random vectors \((X^{i}_m)_{m \geq 1}\) s.t.
  \[
  \forall n \geq 1, X^{i}_n = I^{i}_n W^{i}_0 + (1 - I^{i}_n) W^{i}_n,
  \]
  - where the \((W^{i}_n)_{n \geq 0}\) are i.i.d. r.v.'s with cdf \(F^{i}_W \in \mathcal{R}_{-\alpha^i}\),
  - the \((I^{i}_n)_{n \geq 1}\) are i.i.d. Bernoulli r.v.'s with parameter \(p^{i} \in [0, 1]\),
  - the \(W^{i}_n, n \geq 0\) are independent from the \(I^{i}_k, k \geq 1\)
  - and the \(W^{i}_n, n \geq 0\) and \(I^{i}_k, k \geq 1\) are independent from a Poisson process \(N^{i}(t)\) with parameter \(\lambda^{i}\).

- Define the \(J\) independent processes (1 \leq \(i\) \leq \(J\)) as
  \[
  Y^{i}(t) = c^{i} t - \sum_{m^{i}=1}^{N^{i}(t)} X^{i}_{m^{i}}
  \]
Correlation crisis

Let $T_p$ be the instant of the $p^{th}$ jump of the process $J_t$, and define $(R(t))_{t \geq 0}$ by

$$R(t) = u + \sum_{p \geq 1} \sum_{1 \leq i \leq n} (Y^i(T_p) - Y^i(T_{p-1})) 1\{J_{T_p-1} = i, T_p \leq t\}$$

$$+ \sum_{p \geq 1} \sum_{1 \leq i \leq n} (Y^i(t) - Y^i(T_{p-1})) 1\{J_{T_p-1} = i, T_{p-1} \leq t < T_p\}.$$

A typical modulated risk process with two states (red and blue).
Correlation crises: case where $\alpha^1 < \alpha^i$ for all $i \geq 2$

**Theorem**

As $u \to +\infty$, we have for any $t > 0$

$$
\psi(u, t) \sim \left( \sum_{i=1}^{J} \pi_0(i) E \left( M_i^\perp + \left[ \left[ M_i^{\text{com}} \alpha^1 \right] \right] \right) \right) \bar{F}^1(u),
$$

- where $W_i^1(t)$ is the time spent by the environment process in state 1 during $[0, t]$ given that $J(0) = i$,
- $M_i^\perp$ follows a mixed Poisson distribution with random parameter $\lambda^1(1 - p^1)W_i^1(t)$,
- and $M_i^{\text{com}}$ follows a mixed Poisson distribution with random parameter $\lambda^1 p^1 W_i^1(t)$. 
Correlation crises: case where $\alpha^1 < \alpha^i$ for all $i \geq 2$

**Theorem**

*If besides $\alpha^1 \in \mathbb{N}$, we have*

$$E(M^1_i) = \lambda^1(1 - p^1)E[W^1_i(t)] = \lambda^1(1 - p^1)D_i^1(1, t),$$

*and*

$$E\left([M^\text{com}_i]^\alpha^1\right) = E\left[\sum_{k=0}^{\alpha^1} S(\alpha^1, k)(\lambda^1 p^1)^k(W_i^1(t))^k\right],$$

$$= \sum_{k=0}^{\alpha^1} S(\alpha^1, k)(\lambda^1 p^1)^k D_i^1(k, t),$$

*where $S(\alpha^1, k)$ is the $(\alpha^1, k)$ Stirling number of the second kind, and where for $m \geq 1$, $D_i^1(m, t) = E\left[(W_i^1(t))^m \mid J(0) = i\right]$ Castella et al. (2007)*

*is the $i^{th}$ component of vector $D^1(m, t)$ defined by $D^1(0, t) = 1$ and for $m \geq 1$,

$$D^1(m, t) = r \int_0^t e^{Q(t-u)}A_{11}D^1(m - 1, u)du$$

*where $A_{11}$ is $J \times J$ with coeff. $\delta_{i1}\delta_{j1}$. 
Pure correlation crises: case where $\alpha^i = \alpha \in \mathbb{N} \setminus \{0\}$ for all $1 \leq i \leq 3$

Assume that

- in state 1, the $W_n^1, n \geq 1$ are i.i.d.,
- in state 2, there is a light correlation: the $W_n^2, n \geq 1$ have Gaussian copulas,
- and in state 3, the $W_n^3, n \geq 1$ are given by the basic dependence model with parameter $p_3$. Then,

**Theorem**

*We have for any $t > 0$, as $u \to +\infty$,*

$$
\psi(u, t) \sim \left( \sum_{i=1}^{3} \pi_0(i)[\lambda^1 D^1_i(1, t) + \lambda^2 D^2_i(1, t)] + \lambda^3 (1 - p^3) D^3_i(1, t) + \sum_{k=0}^{\alpha} S(\alpha^3, k)(\lambda^3 p^3)^k D^3_i(k, t) \right) \bar{F}(u).
$$

Similar results may be obtained with classical copulas, and dependence between different state processes. See Biard, Lefèvre and L. (2008) for further details.
1. Ruin and correlation

2. Optimal reserve allocation
   - Optimal allocation problem
   - Heavy-tailed case
   - Light-tailed case

3. Sensitivity and robustness

4. New themes
The allocation problem is the minimization of the risk measure

$$I_T(u_1, u_2) = E \left[ I^1_T(u_1) \right] + E \left[ I^2_T(u_2) \right],$$

for $u_1 \geq 0$ and $u_2 \geq 0$ under the constraint $u_1 + u_2 = u$ for large $u$ where

$$I^i_T(u_i) = \int_0^T 1\{R_i(t)<0\} |R_i(t)| dt \quad i = 1, 2.$$
Heavy-tailed case and $T = \infty$

- No environment process.
- $F_{W_1} \in \mathcal{R}_{-\alpha_1}$ and $F_{W_2} \in \mathcal{R}_{-\alpha_2}$ with $\alpha_1 < \alpha_2$.

**Theorem**

*If we denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ we have for large $u$*

$$
\beta(u) \sim \left( \frac{D'_2}{D'_1} \frac{F_{W_2}(u)}{F_{W_1}(u)} \right)^{1/(\alpha_2-2)},
$$

*where for $i = 1, 2$,*

$$
D'_i = (\alpha_i - 3)^{-1} \frac{1}{c_i} \frac{1}{1 - \psi_i(0)} \frac{\lambda_i + \lambda}{c - (\lambda_i + \lambda)\mu_i} \frac{1}{(\alpha_i - 1)(\alpha_i - 2)(\alpha_i - 3)}.
$$

The finite-time case can also be treated (see Biard, Macci, L. and Veraverbeke (2010) for details).
Light-tailed case and $T = \infty$

- No environment process.
- We assume that the Cramer-Lundberg exponent exists for each process and is equal to $R_i$, $i = 1, 2$ with $R_1 < R_2$.

**Theorem**

For large $u$, the solution is given by

$$u_1 = \frac{R_2}{R_1 + R_2} u + \frac{1}{R_1 + R_2} \log \left( \frac{M'_2}{M'_1} \right) + o(1),$$

$$u_2 = u - u_1 + o(1),$$

where

$$M'_i = -R_i \frac{1}{c_i} \frac{1}{1 - \psi_i(0)} \frac{1 - (\lambda_i + \lambda)\mu_i}{R_i^2((\lambda_i + \lambda)\hat{F}_{W_i}(R_i) - 1)} \quad i = 1, 2.$$ 

Other allocation problems could be considered (see Cénac et al. (2010)).
Ruin and correlation

Optimal reserve allocation

Sensitivity and robustness
- Robustness analysis and ERSM
- Sensitivity analysis of finite-time ruin probabilities

New themes
Motivation

For initial reserve $u \geq 0$ (see L., Mazza and Rullière (2008, 2009)), we show the convergence (in distribution) of the renormalized difference

$$\sqrt{N} \left( \overline{\psi}(u, t) - \overline{\psi}^N(u, t) \right)$$

between the classical finite-time non-ruin probability $\overline{\psi}(u, t)$ and its equivalent obtained when claims are drawn from the empirical distribution of a sample of size $N$ towards a centered Gaussian random variable. The asymptotic variance can be obtained from easy-to-simulate quantities, thanks to so-called partly shifted processes.
OM (or partly shifted) risk processes

Offset-modified or partly shifted risk process (link with Rama Cont’s vertical derivative)

Figure 1. A sample path of the classical risk process $R_t$.

Figure 2. A corresponding sample path of the $x$-partly shifted risk process.
Reliable ruin probability

Set

\[ \phi^Y(u, t) = P \left( R_s^Y \geq 0 \forall s < t \mid R_0^Y = u \right), \]

where \( R_s^Y = R_s - 1\{U < s\} Y \), \( s \geq 0 \), and \( U \) is uniform on \([0, t]\). \( U \), \( Y \) and \( (R_t)_{t \geq 0} \) are mutually independent. Then the variance of the ruin probability is

\[ \text{Var} \left[ \lambda t \phi^Y(u, t) \right]. \]

As Takács’s lemma remains valid for partly shifted processes, those computations can be done quite quickly.
Estimation Risk Solvency Margin (ERSM)

If $u_\eta$ and $u_{\eta,\varepsilon}$ are respectively defined as the initial capital required to ensure that

$$\psi(u_\eta, t) \leq \eta \quad \text{and} \quad \psi_{1-\varepsilon}^{N,\text{reliable}}(u_{\eta,\varepsilon}, t) \leq \eta,$$

the Estimation Risk Solvency Capital $ERSM_{\eta,1-\varepsilon}$ can be defined as the additional capital needed to take estimation risk into account:

$$ERSM_{\eta,1-\varepsilon} = u_{\eta,\varepsilon} - u_\eta.$$

Other estimators may also be used (see Susan Pitts’s recent work).
Because $C^2$ condition is not satisfied, Malliavin calculus on the Poisson space (see Privault and Wei, 2004) cannot be used. We had to use integration by parts to compute the sensitivity of the finite-time ruin probability w.r.t. initial reserve $u \geq 0$ (joint work with Nicolas Privault (2009)).
1 Ruin and correlation
2 Optimal reserve allocation
3 Sensitivity and robustness
4 New themes
   • Longevity risk
   • Other perspectives
**FIGURE:** Migrations and life expectation
Figure 4 Log-mortality structure of French male, 1962-2000
Kappa(t) in the Lee-Carter model (French nat. pop.)

\[
\log \mu_{x,t} = \alpha_x + \beta_x \cdot \kappa_t + \varepsilon_{x,t}, \quad \varepsilon_{x,t} \sim \mathcal{N}(0, \sigma),
\]
Longevity improvements at different ages

Figure 3.a: $b_x$ schedule for the entire time series 1947-1999 and two selected subsamples 1947-1970 and 1976-1999, female

Examining Structural Shifts in Mortality Using the Lee-Carter Method
Lawrence R. Carter, Alexia Prskawetz
**Pure Longevity Risk**

- change of the average trend
- short-term oscillations around the average trend (risk of over-reactions)
- Heterogeneity and basis risk: the evolution of the policyholders mortality is usually different from that of the national population (selection effects).

**Financial Risk**

- Long term interest rate risk
- Counterparty risk
Other perspectives

- s-convex extrema for $t$-monotone distributions
- in ruin theory (univariate and multivariate issues)
- about variable annuities
- Climate change risk and actuarial science (MIRACCLE project)
- Enterprise Risk Management
- Solvency II; accelerating nested simulations
- Value of insurance portfolios