On Pareto optimal allocations for multi-period risks

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Introduction and motivation

Development of new financial products

A new type of assets:

- Recent introduction of a new type of financial contracts with a non-financial underlying risk: ”cat-bonds”, weather derivatives, insurance-linked derivatives...
- Illiquid instruments, with an underlying asset which is not traded on financial markets.
- More recently, cession of part of the bank’s global portfolio to some hedge funds.

Convergence and interplay between finance and insurance:

⇒ Use of the knowledge of financial risk management to the management of other kinds of risk.
⇒ Use of the insurance technology to design structured products.
Main questions

- From a financial point of view:
  - What is the *pricing rule*?
  - What is the *hedging* strategy?

- From an insurance point of view:
  What is the ”*optimal” structure to issue or ”optimal” transfer of a non-tradable *risk* so that a transaction occurs?
Relationship with group diversification

- **Diversification** = Spreading the overall portfolio over a variety of exposures (products, markets, subsidiaries...).
- *Shifting of the focus* from the capital requirements for individual businesses to a group level.
- Key issue in the preparation of the new *Solvency II* framework for the regulation of European insurance undertakings (Pillar 1 on quantitative requirement).

⇒ Optimal allocation of the risk(s) among different various businesses.

Remarks:
- ★ Needs for a perfect fungibility of the capital between business units.
- ★ Needs for a standardization for capital and risk transfers (see Filipovic and Kupper (2007)).
Our objective

⇒ Existing literature in insurance and finance on Pareto-optimal exchange of risk but for a given maturity date.

⇒ What happens when there is a risk stream occurring at different dates, and when the risk exchange could also occur at different dates?
Agenda

⋆ Pareto-optimal allocations in the one-period case:
   ◦ Basic recalls on risk allocations.
   ◦ Pareto-optimal allocation in the case of expected utilities.

⋆ A general result on Pareto-optimal allocations.

⋆ Pareto-optimal allocations in the multi-period setting.

⋆ Some simple examples.
Pareto-optimal allocations in the one-period case
Pareto-optimal allocations in the one-period case

Framework and definitions

Framework:

- Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a standard probability space and \(T > 0\) a fixed maturity.
- Let \(L\) be a space of random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) collecting all risks of time \(T\).

Two agents \(A\) and \(B\)

- Their preferences about risks on \(L\) are represented by some preference functionals \(U_A\) and \(U_B\) satisfying:

\[
X \succeq_i Y \iff U_i(X) \geq U_i(Y) \quad i = A, B.
\]

Some preference functionals: expected utility, monetary utility function, – convex risk measure, – cash-sub linear risk measure...

- Both agents \(A\) and \(B\) face some individual risks at time \(T\) : \(\hat{X}^A\) and \(\hat{X}^B\).

\(\hat{X} = \hat{X}^A + \hat{X}^B\) is the aggregated risk.
About risk allocations:

* An allocation of \( \hat{X} \) between \( A \) and \( B \) is a pair \( (X^A, X^B) \) such that \( X^A + X^B = \hat{X} \). \( A(\hat{X}) \) is the set of all possible allocations.

* An allocation \( (X^A, X^B) \in A(\hat{X}) \) is Pareto-optimal if for any other allocation \( (Y^A, Y^B) \in A(\hat{X}) \), then:

\[
U_A(Y^A) > U_A(X^A) \implies U_B(Y^B) < U_B(X^B)
\]

and

\[
U_B(Y^B) > U_B(X^B) \implies U_A(Y^A) < U_A(X^A).
\]
Case of expected utilities: $U_i(.) = E_P(u_i(.))$ ($i = A, B$)

where $u_i$ is a utility function (i.e. an increasing concave differentiable map $\mathbb{R} \to \mathbb{R}$).

Characterization of Pareto-optimal allocations:

Theorem (Borch, Du Mouchel, Gerber): Let $(X^A, X^B) \in A(\hat{X})$. Then the following statements are equivalent:

1. $(X^A, X^B)$ is Pareto optimal;

2. $(X^A, X^B)$ solves the sup-convolution problem for some constant $\lambda > 0$:

$$[U_A \Box \lambda U_B](\hat{X})) \triangleq \sup_{(Y^A, Y^B) \in A(\hat{X})} \{U_A(Y^A) + \lambda U_B(Y^B)\}.$$ 

3. The equality

$$\frac{u_A'(X_A)}{u_B'(X_B)} = \lambda$$

holds for some constant $\lambda > 0$. 

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A general result
on Pareto-optimal allocations
A more general setting

Framework

Idea: We now consider more general preference functionals $U_i$, defined on a real vector space $\mathcal{L}$ (e.g. space of random variables, processes, real vectors...).

More precisely: All the notions previously introduced are formally generalized. In particular, an element of $\mathcal{L}$ is interpreted as a ”risk”.

★ The initial ”risk” of both agents $A$ and $B$ are $\hat{X}^A \in \mathcal{L}$ and $\hat{X}^B \in \mathcal{L}$. The aggregated ”risk” is $\hat{X} = \hat{X}^A + \hat{X}^B$.

★ To assess their risks, both agents have general preference functionals $U_A, U_B$, now defined on $\mathcal{L}$.

★ The notion of Pareto-optimal allocation is also extended.

Two natural conditions are imposed on the preference functionals $U_i$:

1. Non satiation Property ($\mathcal{U}$): $U_i$ does not attain its supremum on $\mathcal{L}$.
2. Concavity of $U_i$ to translate risk aversion.
Pareto-optimal allocations

**Theorem**: Assume that $U_A$ and $U_B$ are two concave preference functionals, satisfying (U). An allocation is Pareto optimal if and only if it solves the sup-convolution problem:

$$(U_A \Box \lambda U_B)(\hat{X}) \text{ for some } \lambda > 0.$$  

**Comments**:  
⇒ Partial generalization of Borch-Du Mouchel-Gerber’s Theorem for expected utilities (equivalence between 1. and 2., obtained by Gerber (1978)).  
⇒ Extension of results on coherent risk measures (Delbaen (2000)), on convex risk measures (B. and El Karoui (2005)) and on monetary utility functions (Jouini, Schachermayer and Touzi (2006)).
Pareto-optimal allocations in the multi-period case
Pareto-optimal allocations in the multi-period case

The risks are now spread over \( N \geq 2 \) future dates \( T_1, T_2, \ldots, T_N \).

Framework and notations

★ Let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)) \) be a filtered probability space and \( L^0_n \) be the space of \( \mathcal{F}_n \)-measurable random variables, describing the risks occurring at time \( T_n \). The space of risk streams is then

\[
\mathbf{L} \subseteq \{ \mathbf{X} = (X_n)_{1 \leq n \leq N} : X_n \in L^0_n, \forall n \}.
\]

The agents:

★ The initial risk streams are:

\[
\mathbf{X}^A = (\hat{X}^A_n)_{1 \leq n \leq N} \in \mathbf{L} \quad \text{and} \quad \mathbf{X}^B = (\hat{X}^B_n)_{1 \leq n \leq N} \in \mathbf{L}.
\]

The aggregated risk stream is \( \mathbf{X} = \mathbf{X}^A + \mathbf{X}^B \).

★ To assess their risks, both agents have general concave preference functionals \( U_A, U_B \), defined on \( \mathbf{L} \), and satisfying the non satiation Property (\( \mathfrak{U} \)).
Characterization of Pareto-optimal allocations

Corollary: Assume that $U_A$ and $U_B$ are two concave preference functionals, satisfying $(\mathcal{U})$. An allocation is Pareto optimal if and only if it solves the sup-convolution problem $(U_A \boxplus \lambda U_B)(\hat{X})$ for some $\lambda > 0$.

Going further?

⋆ One-period case: The Borch-Du Mouchel-Gerber’s Theorem gives the equivalence between:

1. Pareto optimality,
2. $\lambda$ sup-convolution,
3. constant ratio of marginal utilities.

⋆ Multi-period case: We have the equivalence between:

1. Pareto optimality,
2. $\lambda$ sup-convolution.

The idea is to find an "equivalent" statement for 3. To do so, we consider here expected utilities for processes.
Pareto-optimal allocations in the case of expected utilities for processes

Framework

★ A utility function for processes is a differentiable map $w : \mathbb{R}^N \to \mathbb{R}$ that is concave and strictly increasing in each variable (i.e $\nabla w > 0$).

★ An expected utility for processes is any preference functional $U : \mathcal{L} \to \mathbb{R}$ of the type:

$$U(X_1, \ldots, X_N) \triangleq \mathbb{E}_\mathbb{P} w(X_1, \ldots, X_N),$$

for some utility function for processes, $w$.

$\Rightarrow$ We already know that an allocation is Pareto optimal if and only if it solves the sup-convolution problem (previous corollary), $(U_A \Box \lambda U_B)(\hat{X})$ for some $\lambda > 0$.

$\Rightarrow$ We want to relate Pareto optimality with the notion of ”marginal utilities” $\frac{\partial w}{\partial x_n}$. 

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Pareto-optimal allocation for bounded risks

We assume that \( L = L^\infty \).

It is possible to obtain a full generalization of Borch-Du Mouchel-Gerber Theorem (equivalence between 1. and 3.).

**Theorem:** A bounded allocation \((X^A, X^B)\) is Pareto optimal if and only if there exists a constant \( \lambda > 0 \) such that for any \( n = 1, \ldots, N \):

\[
\frac{\mathbb{E} \left[ \frac{\partial w_A}{\partial x_n} (X^A) \big| \mathcal{F}_n \right]}{\mathbb{E} \left[ \frac{\partial w_B}{\partial x_n} (X^B) \big| \mathcal{F}_n \right]} = \lambda.
\]
Pareto-optimal allocation for more general risks

We assume that \( L^\infty \subset L \).

\( \star \) Under a measurability condition: \( \frac{\partial w_i}{\partial x_n}(X^i) \in \mathcal{F}_n, \forall n, i = A, B. \)

**Theorem:** Under this measurability condition, an allocation \((X^A, X^B)\) is Pareto optimal if and only if there exists a constant \( \lambda > 0 \) such that, for any \( n = 1, ..., N \)

\[
\frac{\partial w_A}{\partial x_n}(X^A) = \lambda.
\]

\( \star \) Under an integrability condition: \( \frac{\partial w_i}{\partial x_n}(X^i) \in L^1, \forall n, i = A, B. \)

**Theorem:** Under the integrability condition, if an allocation \((X^A, X^B)\) is Pareto optimal, then there exists a constant \( \lambda > 0 \) such that, for any \( n = 1, ..., N \)

\[
\mathbb{E} \left[ \frac{\partial w_A}{\partial x_n}(X^A)|\mathcal{F}_n \right] = \lambda.
\]
Some simple examples
Some examples of Pareto-optimal allocations

We want to characterize Pareto-optimal allocations in different cases, involving expected utilities for processes, for *bounded* risks and *two* dates.

**Time-additive expected utilities**

\[ U_i(X_1, X_2) \triangleq \mathbb{E}(u_i(X_1)) + \beta_i \mathbb{E}(u_i(X_2)), \quad \text{for } i = A, B \]

for some utility functions \( u_i \) and decay factors \( \beta_i > 0 \).

**Characterization of Pareto optimal allocations :**

**Proposition :** An allocation \((X^A, X^B)\) is Pareto optimal if and only if there exists some \( \lambda > 0 \) such that:

\[
\frac{u'_A(X^A_1)}{u'_B(X^B_1)} = \lambda \quad \text{and} \quad \frac{u'_A(X^A_2)}{u'_B(X^B_2)} = \lambda \frac{\beta_B}{\beta_A}.
\]
Illustration: the exponential utility

\[ U_i(X_1, X_2) \triangleq \mathbb{E}(u_i(X_1)) + \beta_i \mathbb{E}(u_i(X_2)), \quad \text{where } u_i(x) = -\exp(-\gamma_i x). \]

An allocation \((X^A, X^B)\) is Pareto optimal if and only if:

\[
X_1^A = \frac{\gamma B}{\gamma A + \gamma B} \hat{X}_1 - \frac{1}{\gamma A + \gamma B} \ln(c(\gamma_A, \gamma_B))
\]
\[
X_2^A = \frac{\gamma B}{\gamma A + \gamma B} \hat{X}_2 - \frac{1}{\gamma A + \gamma B} \ln(c(\gamma_A \beta_A, \gamma_B \beta_B)),
\]

for some deterministic function \(c\).

⇒ Proportional and identical risk sharing at both dates.
⇒ No impact of the decay factor on the risk sharing rule.
⇒ However, specific role of the decay factor on the correction term.
Non time-additive expected utilities

\[ U_i(X_1, X_2) \triangleq \mathbb{E}(u_i(X_1 + \beta_i X_2)), \quad \text{for } i = A, B \]

for some utility functions \( u_i \) and decay factors \( \beta_i > 0 \).

Characterization of Pareto optimal allocations:

Proposition: i) If \( \beta_A \neq \beta_B \) then there is no Pareto optimal allocation.

ii) If \( \beta_A = \beta_B \triangleq \beta \), then \((X^A, X^B)\) is Pareto optimal if and only if there exists some \( \lambda > 0 \) such that :

\[
\frac{u_A'(X_1^A + \beta X_2^A)}{u_B'(X_1^B + \beta X_2^B)} = \lambda.
\]
Illustration: the exponential utility

\[ U_i(X_1, X_2) \triangleq \mathbb{E}(u_i(X_1 + \beta_i X_2)), \text{ where } u_i(x) = -\exp(-\gamma_i x). \]

An allocation is Pareto optimal if and only if:

\[ X_1^A + \beta X_2^A = \frac{\gamma_B}{\gamma_A + \gamma_B} (\hat{X}_1 + \beta \hat{X}_2) + c, \]

for some constant \( c \).

\( \Rightarrow \) Proportional risk sharing rule of the aggregated risks of time 1 and time 2.

\( \Rightarrow \) Possible dependency on the initial risk allocations.

\( \Rightarrow \) Possible inter-temporal risk exchanges.
Some concluding comments

★ So far :

◇ a general result to characterize Pareto-optimal allocations of risk, which can be applied to a multi-period setting,

◇ necessary and sufficient conditions in terms of ”conditional marginal utilities” in the framework of expected utilities for processes,

◇ explicit characterization of Pareto-optimal allocation rules in some cases.

★ Still under work :

◇ conditions in terms of ”marginal utilities” in a more general framework,

◇ and other explicit examples....

★ Extensions ?

◇ More than two agents?

◇ Continuous time?

◇ Inferring the preference functional from existing transactions?
Some references...

- **F. Delbaen**: Coherent risk measures, Pisa Lecture Notes (2000).